On Zudilin's q-question about Schmidt's problem

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Abstract. We propose an elemantary approach to Zudilin's q-question about Schmidt's problem [Electron. J. Combin. 11 (2004), #R22], which has been solved in a previous paper [Acta Arith. 127 (2007), 17–31]. The new approach is based on a q-analogue of our recent result in [J. Number Theory 132 (2012), 1731–1740] derived from q-Pfaff-Saalschütz identity.

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1 Introduction

In 2007, answering a question of Zudilin [7], the following result was proved in [3].

Theorem 1.1. *Let $r \geqslant 1$. Then there exists a unique sequence of polynomials $\{c_i^{(r)}(q)\}_{i=0}^{\infty}$ in q with nonnegative integral coefficients such that, for any $n \geq 0$,*

$$\sum_{k=0}^{n} q^{r\binom{n-k}{2}+(1-r)\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}^r \begin{bmatrix} n+k \\ k \end{bmatrix}^r = \sum_{i=0}^{n} q^{\binom{n-i}{2}+(1-r)\binom{i}{2}} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} c_i^{(r)}(q). \tag{1.1}$$

Here, the q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q)_n}{(q)_k (q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise,} \end{cases}$$

where $(q)_0 = 1$ and $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$ for $n = 1, 2, \dots$. It is well known that $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial in q with nonnegative integral coefficients of degree $k(n-k)$ (see [2, p. 33]).

The proof of (1.1) given in [3] is a q-analogue of Zudilin's [7] approach to Schmidt's problem (see [5,6]) by first using the q-Legendre inversion formula to obtain a formula for $c_k^{(r)}(q)$ and then applying a basic hypergeometric identity due to Andrews [1] to show that

the latter expression is indeed a polynomial in q with nonnegative integral coefficients. In this paper we propose a new and elementary approach to Zudilin's q-question, which yields not only a new proof of Theorem 1.1, but also more solutions to Zudilin's q-question about Schmidt's problem.

Our starting point is the following q-version of Lemma 4.2 in [4].

Lemma 1.2. *Let $k \geqslant 0$ and $r \geqslant 1$. Then there exists a unique sequence of Laurent polynomials $\{P_{k,i}^{(r)}(q)\}_{i=k}^{rk}$ in q with nonnegative integral coefficients such that, for any $n \geqslant k$,*

$$\begin{bmatrix} n \\ k \end{bmatrix}^r \begin{bmatrix} n+k \\ k \end{bmatrix}^r = \sum_{i=k}^{\min\{n,rk\}} q^{(rk-i)n} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} P_{k,i}^{(r)}(q). \tag{1.2}$$

Moreover, the polynomials $P_{k,i}^{(r)}(q)$ can be computed recursively by $P_{k,k}^{(1)}(q) = 1$ and

$$P_{k,k+j}^{(r+1)}(q) = \sum_{i=k}^{rk} q^{(j-i)(j+k)} \begin{bmatrix} k+i \\ i \end{bmatrix} \begin{bmatrix} k \\ i-j \end{bmatrix} \begin{bmatrix} k+j \\ j \end{bmatrix} P_{k,i}^{(r)}(q), \ 0 \leqslant j \leqslant rk. \tag{1.3}$$

To derive Theorem 1.1 from Lemma 1.2 we first consider a more general problem. Let $f(x,y)$ and $g(x,y)$ be any polynomials in x and y with integral coefficients. Multiplying (1.2) by $q^{-nkr+f(k,r)}$ and summing over k from 0 to n we obtain

$$\sum_{k=0}^{n} q^{-nkr+f(k,r)} \begin{bmatrix} n \\ k \end{bmatrix}^r \begin{bmatrix} n+k \\ k \end{bmatrix}^r = \sum_{i=0}^{n} q^{-ni-g(i,r)} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} \sum_{k=0}^{i} T_{k,i}^{(r)}(q), \tag{1.4}$$

where

$$T_{k,i}^{(r)}(q) = q^{f(k,r)+g(i,r)} P_{k,i}^{(r)}(q), \ 0 \leqslant k \leqslant i, \text{ and } P_{k,i}^{(r)}(q) = 0 \text{ if } i > kr. \tag{1.5}$$

Obviously $T_{k,i}^{(r)}(q)$ are Laurent polynomials in q with nonnegative integral coefficients. For example, taking $f = g = 0$, we immediately obtain the following result.

Theorem 1.3. *Let $r \geqslant 1$. Then there exists a unique sequence of Laurent polynomials $\{b_i^{(r)}(q)\}_{i=0}^{\infty}$ in q with nonnegative integral coefficients such that, for any $n \geq 0$,*

$$\sum_{k=0}^{n} q^{-rkn} \begin{bmatrix} n \\ k \end{bmatrix}^r \begin{bmatrix} n+k \\ k \end{bmatrix}^r = \sum_{i=0}^{n} q^{-ni} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} b_i^{(r)}(q). \tag{1.6}$$

Moreover, we have $b_i^{(r)}(q) = \sum_{k=0}^{i} P_{k,i}^{(r)}(q)$.

Now, we look for a sufficient condition for $T_{k,i}^{(r)}(q)$ in (1.4) to be a polynomial. It follows from (1.3) that

$$T_{k,i}^{(r+1)}(q) = \sum_{j=k}^{rk} q^{A} \begin{bmatrix} k+j \\ j \end{bmatrix} \begin{bmatrix} k \\ i-j \end{bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} T_{k,j}^{(r)}(q), \tag{1.7}$$

where

$$A = f(k, r+1) + g(i, r+1) - f(k, r) - g(j, r) + i(i - k - j). \tag{1.8}$$

Hence, the positivity of A will ensure that $T_{k,i}^{(r)}(q)$ is a polynomial in q.

We shall first prove Lemma 1.2 in the next section and then prove Theorem 1.1 in Section 3 by choosing special polynomials f and g. Some open problems are raised in Section 4.

2 Proof of Lemma 1.2

We proceed by induction on r. We need the following form of Jackson's q-Pfaff-Saalschütz identity (see [2, pp. 37-38] or [5] for example):

$$\begin{bmatrix} m+n \\ M \end{bmatrix} \begin{bmatrix} n \\ N \end{bmatrix} = \sum_{j \geqslant 0} q^{(N-j)(M-m-j)} \begin{bmatrix} M-m \\ j \end{bmatrix} \begin{bmatrix} N+m \\ m+j \end{bmatrix} \begin{bmatrix} m+n+j \\ M+N \end{bmatrix}. \tag{2.1}$$

Substituting $m \to k-i$, $n \to n+i$, $M \to n-i$ and $N \to i$ in (2.1), we get

$$\begin{bmatrix} n+k \\ n-i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} = \sum_{j=0}^{i} q^{(i-j)(n-k-j)} \begin{bmatrix} n-k \\ j \end{bmatrix} \begin{bmatrix} k \\ i-j \end{bmatrix} \begin{bmatrix} n+k+j \\ n \end{bmatrix},$$

which can be rewritten as

$$\begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} n+i \\ i \end{bmatrix} = \sum_{i=0}^{i} q^{(i-j)(n-k-j)} \frac{(q)_{k+i}(q)_j}{(q)_{k+j}(q)_i} \begin{bmatrix} k \\ i-j \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} \begin{bmatrix} n+k+j \\ j \end{bmatrix}. \tag{2.2}$$

It is clear that (1.2) holds for $r = 1$ with $P_{k,k}^{(r)}(q) = 1$. Suppose that (1.2) holds for some $r \geqslant 1$. Multiplying both sides of (1.2) by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]\left[\begin{smallmatrix} n+k \\ k \end{smallmatrix}\right]$ and applying (2.2), we immediately get

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}^{r+1} \begin{bmatrix} n+k \\ k \end{bmatrix}^{r+1} &= \sum_{i=k}^{rk} q^{(rk-i)n} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+k \\ k \end{bmatrix} P_{k,i}^{(r)}(q) \\ &\quad \times \sum_{j=0}^{i} q^{(i-j)(n-k-j)} \frac{(q)_{k+i}(q)_j}{(q)_{k+j}(q)_i} \begin{bmatrix} k \\ i-j \end{bmatrix} \begin{bmatrix} n-k \\ j \end{bmatrix} \begin{bmatrix} n+k+j \\ j \end{bmatrix} \\ &= \sum_{j=0}^{rk} q^{(rk-j)n} \begin{bmatrix} n \\ k+j \end{bmatrix} \begin{bmatrix} n+k+j \\ k+j \end{bmatrix} P_{k,k+j}^{(r+1)}(q), \end{aligned} \tag{2.3}$$

where $P_{k,k+j}^{(r+1)}(q)$ is given by (1.3). By the induction hypothesis, these $P_{k,k+j}^{(r+1)}(q)$ are Laurent polynomials in q with nonnegative integral coefficients. Hence Lemma 1.2 is true for $r+1$.

3 Proof of Theorem 1.1

In (1.4), taking $f(k,r)=r\binom{k+1}{2}$, $g(i,r)=(r-2)\binom{i}{2}-i$, and multiplying by $q^{\binom{n}{2}}$, we obtain (1.1) with

$$c_i^{(r)}(q)=q^{(r-2)\binom{i}{2}-i}\sum_{k=0}^{i}q^{r\binom{k+1}{2}}P_{k,i}^{(r)}(q). \tag{3.1}$$

By (1.8) the corresponding A reads as follows

$$A=(r-2)\left[\binom{i}{2}-\binom{j}{2}\right]+\binom{i-k}{2}+(i-1)(i-j).$$

If $r\geqslant 2$, since $i\geqslant j$, we have $A\geqslant 0$. If $r=1$, then (1.7) implies that $j=k$ and $A=2\binom{i-k}{2}\geqslant 0$. Thus the $c_i^{(r)}(q)$ in (3.1) is a polynomial in q. For example, by (1.5) we have

$$T_{k,i}^{(2)}(q)=q^{2\binom{i-k}{2}}\begin{bmatrix}2k\\ i\end{bmatrix}\begin{bmatrix}i\\ k\end{bmatrix}^2,$$

and

$$c_i^{(2)}(q)=\sum_{k=0}^{i}q^{2\binom{i-k}{2}}\begin{bmatrix}2k\\ i\end{bmatrix}\begin{bmatrix}i\\ k\end{bmatrix}^2,$$

which coincides with [3, (3,1)].

4 Open problems

For any positive integers r and s, it is easy to see that there are uniquely determined rational numbers $c_k^{(r,s)}$ $(k\geqslant 0)$, independent of n $(n\geqslant 0)$, satisfying

$$\sum_{k=0}^{n}\binom{n}{k}^r\binom{n+k}{k}^r=\sum_{k=0}^{n}\binom{n}{k}^s\binom{n+k}{k}^s c_k^{(r,s)}. \tag{4.1}$$

When $s=1$ and $r\geqslant 1$, the integrality of $c_k^{(r,s)}$ is the original problem of Schmidt [5]. When $s>1$ and $r>s$, we observe that the numbers $c_k^{(r,s)}$ are not always integers. From arithmetical point of view, the following problems may be interesting.

Conjecture 4.1. *For any $s>1$ and $n\geqslant 0$, there is an integer $r>s$ such that all the numbers $c_k^{(r,s)}$ $(0\leqslant k\leqslant n)$ are integers.*

For $s=2$, via Maple, we find that the least such integers $r:=r(n,s)$ are $r(0,2)=r(1,2)=r(2,2)=3, r(3,2)=7, r(4,2)=32, r(5,2)=212$.

Conjecture 4.2. *For any $r>s>1$, there is a positive integer n such that $c_n^{(r,s)}$ is not an integer.*

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